problem session 5, part 1

22 p 149

Problem Session (Chapter 6)

Section 6.1

22 p149 I, J ideals in a ring R

$K = \{ab \mid a \in I, b \in J\} \subset R$ may be not an ideal in R

Example $R = \mathbb{Z}[x]$ $I = J = (2, x)$ - the set of all polynomials from $\mathbb{Z}[x]$ with even constant terms

$K \ni 4$ $K \ni x^2$, however $K \not\ni 4 + x^2$ (K is not even a subring of R)

$4 + x^2$ is not a product of two polynomials with integral coefficients (and even constant terms)

$4 + x^2 \in \mathbb{Z}[x] \subset \mathbb{Q}[x]$

$4 + x^2 \in \mathbb{Q}[x]$ is irreducible, therefore does not factor into a product of two polynomials of positive degree in $\mathbb{Q}[x]$, therefore in $\mathbb{Z}[x]$

$\deg(4 + x^2) = 2$

The only possible presentation

$4 + x^2 = ab$

happens when $a \in \mathbb{Z}$ or $b \in \mathbb{Z}$; looking at the leading term of x^2, we conclude that the integer must be $\pm 1 \notin I, J$.

Rem In the rings $\mathbb{Z}$, $F[x]$ (F - a field) every ideal is principal.

Prop If the ring R is commutative, and with identity $I=(x)$, $J=(y)$ are principal, then $K=\{ab \mid a\in I, b\in J\}$ is an ideal

Pf $I=(x)=\{xr_1 \mid r_1\in R\}$ $\quad J=(y)=\{yr_2 \mid r_2\in R\}$

$$K=\{xr_1 yr_2 \mid r_1, r_2\in R\}=\{xy r_1r_2 \mid r_1,r_2\in R\}=\{xyr \mid r\in R\}=(xy)$$